Cauer Ladder Network Representation of Eddy-Current Fields for Model Order Reduction Using Finite Element Method

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A novel model order reduction method for efficient analysis of eddy current problems is proposed in which the electromagnetic fields are decomposed into a sequence of electric and magnetic field modes and the field equation is represented by the Cauer ladder network. The modes are calculated sequentially by electric and magnetic computations using the conventional finite element method. The network equation is applicable in a wide frequency range and can be solved in frequency as well as in time domains with minimum computational loads. In this digest, the formulation of the method and a numerical test application are presented.

***Index Terms*— Cauer ladder network, eddy current, finite element method, model order reduction.**

I. Introduction

Model order reduction (MOR) methods have been recently gaining ground in the area of electromagnetic field analysis arising in the design of electric machines and electromagnetic devices. They offer the potential to drastically reduce the unacceptable computation loads that the finite element method (FEM) entails [1]. The MOR methods are also important in homogenization methods for electromagnetic fine structures. This paper proposes a novel and effective MOR method in which the linear eddy current fields are represented by the Cauer ladder network (CLN).

We have been investigating the equivalent circuit representation approaches using the CLN in the homogenization of laminated magnetic sheets [2], [3]. By the homogenization method, more effective and accurate computations would become possible in the analysis of electric machines with laminated iron cores. It has been shown that wide frequency ranges are covered by minimum circuit elements by the CLN representation. The frequency range is determined specifically and can be extended by adding elements hierarchically. This approach was also applied to the electromagnetic field analysis in cylinders and spheres [4]. The CLN representation was noted to be quite general in the realm of eddy current field. Using the CLN, this paper realizes a simple MOR method that can reconstruct the eddy current field far more efficiently than the conventional FEM method.

II. Representation of Cauer Ladder Network

We consider an analysis domain governed by Maxwell's equations of eddy current fields as shown in Fig. 1. The domain includes magnetic and conductive materials, where the permeability μ [H/m] and the conductivity σ [S/m] are constant in time. A power supply with single output energizes the domain driving voltage v [V] and current i [A].

The eddy current fields are generally characterized by impedance $Z(s)$ in the frequency domain where $s = \mathrm{j}\omega$. The impedance, being a rational function, can be expressed as a continued fraction using the Euclidean algorithm as follows:

$$Z(s) = \frac{v(s)}{i(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \cdots}{a_0 + a_1 s + a_2 s^2 + \cdots}$$
$$= R_0 + \frac{1}{1/sL_1 +}\ \frac{1}{R_2 +}\ \frac{1}{1/sL_3 +}\ \frac{1}{R_4 +}\ \frac{1}{1/sL_5 +}\ \cdots. \tag{1}$$

The impedance can be realized by the CLN shown in Fig. 2, from the continued fraction of (1). This suggests that the field equations in the analysis domain is equivalent to the equations of the network. The network is characterized by resistances (R_0, R_2, R_4,...) [Ω] and inductances (L_1, L_3, L_5,...) [H]. The voltages across the resistors and currents through the inductors are represented by e_{2n} [V] and h_{2n+1} [A], respectively. The infinite sequence of the resistors and inductors can be truncated for a desired accuracy, with the terminating element being R_{2N} (resistive termination) or L_{2N+1} (inductive termination), where N is the number of stages of the CLN [3].

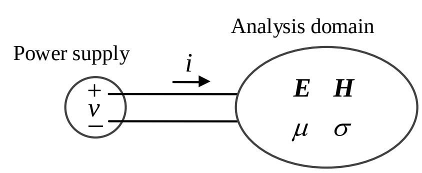

Fig. 1. Analysis domain and power supply.

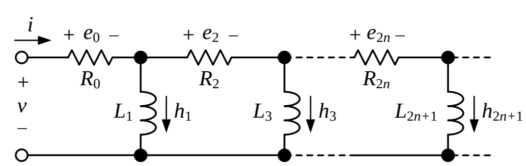

Fig. 2. The Cauer ladder network.

In the ladder network, we notice that there are no couplings between resistors and between inductors. This suggests that the electric and magnetic fields ($\boldsymbol{E}$ [V/m] and $\boldsymbol{H}$ [A/m]) can be expanded by orthogonal basis $\boldsymbol{E}_{2n}$ [m^{-1}] (electric basis functions) and $\boldsymbol{H}_{2n+1}$ [m^{-1}] (magnetic basis functions), respectively, corresponding to e_{2n} and h_{2n+1} as follows:

$$\boldsymbol{E} = \sum_{n=0}^{\infty} e_{2n}\boldsymbol{E}_{2n}\,, \quad \boldsymbol{H} = \sum_{n=0}^{\infty} h_{2n+1}\boldsymbol{H}_{2n+1}\,, \tag{2}$$

$$\int_{\Omega} \sigma \boldsymbol{E}_{2n} \cdot \boldsymbol{E}_{2m} \mathrm{d}V = (1/R_{2n})\delta_{nm}, \tag{3}$$

$$\int_{\Omega} \mu \boldsymbol{H}_{2n+1} \cdot \boldsymbol{H}_{2m+1} \mathrm{d}V = L_{2n+1}\delta_{nm}, \tag{4}$$

where δ_{nm} denotes Kronecker's delta.

The basis functions are recursively related so that the network equations of the CLN are derived as follows:

$$\nabla \times (\boldsymbol{H}_{2n+1} - \boldsymbol{H}_{2n-1}) = R_{2n}\sigma \boldsymbol{E}_{2n}, \tag{5}$$

$$\nabla \times (\boldsymbol{E}_{2n} - \boldsymbol{E}_{2n-2}) = -(1/L_{2n-1})\mu \boldsymbol{H}_{2n-1}, \tag{6}$$

where $\boldsymbol{H}_{-1} = 0$ and $\boldsymbol{E}_0$ is the electric field created by unit voltage (1 V) of the power supply.

Substituting (2) in Maxwell's equations,

$$\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E}, \text{ and } \nabla \times \boldsymbol{E} = -\mathrm{j}\omega\mu \boldsymbol{H}, \tag{7}$$

and using the orthogonalities (3), (4) and the recursive relations (5), (6), one arrives at the following Kirchhoff's current and voltage laws to the CLN:

$$e_0 / R_0 = i, \quad (e_{2n} / R_{2n}) - (e_{2n-2} / R_{2n-2}) = -h_{2n-1}, \tag{8}$$

$$\mathrm{j}\omega L_1 h_1 = -e_0 + v, \quad \mathrm{j}\omega(L_{2n+1}h_{2n+1} - L_{2n-1}h_{2n-1}) = -e_{2n}\,. \tag{9}$$

The procedure of the proposed method is summarized as follows:

Step 0: Initially solve electric field $\boldsymbol{E}_0$ under a given voltage condition. Calculate R_0 by (3).

For $n = 1, 2, 3,\dots,N$**,** do **Step 1** and **2** recursively.

Step 1: Solve magnetic field by

$$\nabla \times \tilde{\boldsymbol{H}}_{2n-1} = R_{2n}\sigma \boldsymbol{E}_{2n-2}, \tag{10}$$

and set $\boldsymbol{H}_{2n-1} = \boldsymbol{H}_{2n-3} + \tilde{\boldsymbol{H}}_{2n-1}$. Calculate L_{2n-1} by (4).

Step 2: Solve electric field by

$$\nabla \times \tilde{\boldsymbol{E}}_{2n} = -(1/L_{2n-1})\mu \boldsymbol{H}_{2n-1}, \tag{11}$$

and set $\boldsymbol{E}_{2n} = \boldsymbol{E}_{2n-2} + \tilde{\boldsymbol{E}}_{2n}$. Calculate R_{2n} by (3).

Step 3: Solve the network in frequency or time domain using (8) and (9).

Step 4: Reconstruct electric and magnetic fields using (2) and calculate quantities like Joule loss and others.

In solving the electric (**Step 0** and **2**) and magnetic fields (**Step 1**), the divergence conditions of Maxwell's equations and appropriate boundary conditions are applied and unique solutions are obtained. The electric fields are solved only in conductive regions and undetermined in the non-conductive regions. Thus, the eddy-current field is decomposed into electric and magnetic field components yielded by a usual finite-element static-field solver.

III. Numerical Application Using FEM

The proposed method is applied to a pair of cylindrical conductors arranged in parallel, such that the conductors carry equal currents but in opposite directions. The diameter of the conductors is 0.85mm, and the distance between the axes of the conductors is 0.95mm. The conductivity of the conductors is $5.8{\times}10^7$ S/m, while the permeability is equal to that of the air. The problem is solved by two-dimensional analysis. The analysis domain is a circular region with 2.0 mm diameter, where Dirichlet boundary condition for the magnetic vector potential is prescribed on the outer boundary.

The CLN of inductive termination is applied where numbers of the stages N are 2 to 5. Fig.3 shows the distributions of current density $\boldsymbol{J}_{2n} = \sigma \boldsymbol{E}_{2n}$ in the conductors, and the distributions of flux density $\boldsymbol{B}_{2n+1} = \mu_0 \boldsymbol{H}_{2n+1}$ in the entire domain related to corresponding resistors and inductors. The impedance of the CLN derived by the proposed method is shown in Fig. 4, together with the result by the conventional FEM. Even the CLN with three stages provides accurate results up to 1 MHz within an error of 1%. Its computation cost is only for three static magnetic field analyses, whereas the conventional FEM requires complex matrix inversions at all the frequencies of interest.

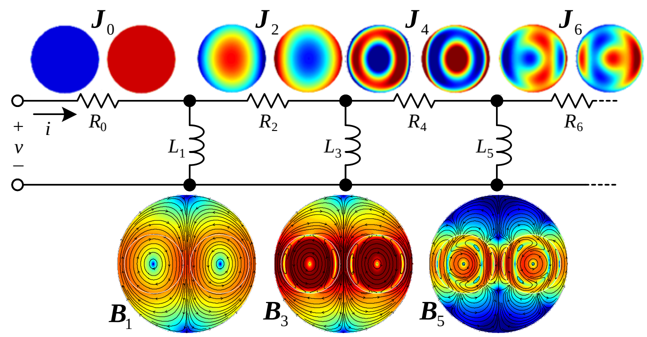

Fig. 3. Electric and magnetic field modes related with CLN.

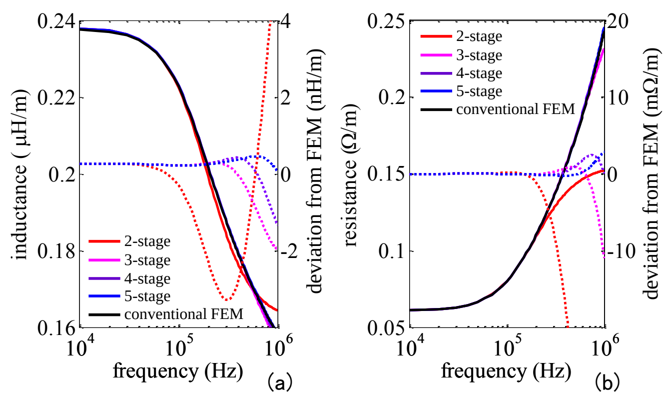

Fig. 4. The impedance of the conductors, (a) inductance and (b) resistance plotted by solid lines, and the differences between the proposed method and the conventional FEM analysis plotted by dashed lines.

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